

WHERE DOES THE L^p -NORM OF A WEIGHTED POLYNOMIAL LIVE?

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ABSTRACT. For a general class of nonnegative weight functions $w(x)$ having bounded or unbounded support $\Sigma \subset \mathbf{R}$, the authors have previously characterized the smallest compact set \mathfrak{S}_w having the property that for every $n = 1, 2, \dots$ and every polynomial P of degree $\leq n$,

$$\|[w(x)]^n P(x)\|_{L^\infty(\Sigma)} = \|[w(x)]^n P(x)\|_{L^\infty(\mathfrak{S}_w)}.$$

In the present paper we prove that, under mild conditions on w , the L^p -norms ($0 < p < \infty$) of such weighted polynomials also “live” on \mathfrak{S}_w in the sense that for each $\eta > 0$ there exist a compact set Δ with Lebesgue measure $m(\Delta) < \eta$ and positive constants c_1, c_2 such that

$$\|w^n P\|_{L^p(\Sigma)} \leq (1 + c_1 \exp(-c_2 n)) \|w^n P\|_{L^p(\mathfrak{S}_w \cup \Delta)}.$$

As applications we deduce asymptotic properties of certain extremal polynomials that include polynomials orthogonal with respect to a fixed weight over an unbounded interval. Our proofs utilize potential theoretic arguments along with Nikolskii-type inequalities.

1. Introduction. In 1974, G. Freud [3] proved the following “infinite-finite range inequality” for weighted polynomials.

Suppose that Q is an even, convex, positive function on \mathbf{R} , differentiable on $(0, \infty)$ and $Q'(t)$ is positive and increasing to ∞ for $0 < t < \infty$. Then there exist positive constants c_1, c_2, c_3 depending only on Q with the following property: For every integer $n \geq 1$ and every polynomial P of degree not more than n ,

$$(1.1) \quad \int_{-\infty}^{\infty} [P(x) \exp(-Q(x))]^2 dx \\
 \leq (1 + c_1 \exp(-c_2 n)) \cdot \int_{|t| \leq c_3 q_{2n}} [P(t) \exp(-Q(t))]^2 dt,$$

where q_{2n} is defined by the equation

$$(1.2) \quad q_{2n} Q'(q_{2n}) = 2n.$$

This inequality has been generalized or investigated in further detail for specific weight functions by several authors including Bonan [1], Lubinsky [7], Zalik [15]

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and the present authors [9, 10]. In [11], we obtained the following sharp result for the sup norm, under less restrictive conditions on Q .

Let $a_n > 0$ be defined by the equation

$$(1.3) \quad \frac{2}{\pi} \int_0^1 \frac{a_n x Q'(a_n x)}{\sqrt{1-x^2}} dx = n.$$

Then for every integer $n \geq 1$ and polynomial P of degree not exceeding n ,

$$(1.4) \quad \max_{x \in \mathbf{R}} |W(x)P(x)| = \max_{|t| \leq a_n} |W(t)P(t)|,$$

where $W(x) := \exp(-Q(x))$. Moreover, (1.4) cannot be improved in the sense that the sequence $\{a_n\}$ cannot be replaced by $\{a_n(1-\delta)\}$ for any positive δ .

In this paper, our aim is to obtain similar precise results for the L^p -norms of the “weighted polynomials,” i.e. expressions of the form $W(x)P(x)$, where W is a weight function and P is a polynomial. Our theorems are general in that they apply to weights W with bounded or unbounded support (not necessarily an interval) and allow W to have zeros at interior points. Of particular interest are the cases when W is supported on \mathbf{R} , $[0, \infty)$, or on a finite union of disjoint closed intervals. In our investigations, we also obtain new results concerning the L^∞ -norm of weighted polynomials that complement those in [11].

In the next section we state and discuss our main results. The proofs are given in §3.

2. Main results. We begin by recalling some definitions and theorems that appear in [11].

DEFINITION 2.1. Let $w: \mathbf{R} \rightarrow [0, \infty)$. We say that w is an *admissible weight function* if each of the following properties holds.

- (i) $\Sigma := \text{supp}(w)$ has positive capacity.
- (ii) $Z := \{x \in \Sigma: w(x) = 0\}$ has capacity zero.
- (iii) The restriction of w to Σ is continuous on Σ .
- (iv) If Σ is unbounded, then $|x|w(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $x \in \Sigma$.

By $\text{supp}(w)$ we mean the closure of the set where $w > 0$ and by *capacity* we mean the inner logarithmic capacity (cf. [14, p. 55]). We use $C(E)$ to denote the capacity of a set $E \subset \mathbf{R}^2$. The class of all polynomials of degree at most n is denoted by Π_n . We also adopt the convention that c, c_1, c_2 , etc. will denote positive constants that are independent of n , but may depend on w and other relevant parameters. Furthermore, the same symbol may denote different values even within a single formula. Constants that retain their values will be denoted by capital letters.

If K is a compact set with positive capacity, then ν_K will denote the unique unit equilibrium measure on K with the property that

$$(2.1) \quad \int_K \log |x-t| d\nu_K(t) = \log C(K)$$

quasi-everywhere (q.e.) on K (cf. [14, p. 60]). A property is said to hold q.e. on a set A if the subset $E \subset A$ where it does not hold satisfies $C(E) = 0$.

For an admissible weight w , we always set

$$(2.2) \quad Q(x) := \log[1/w(x)].$$

Finally, if $K \subset \Sigma \setminus Z$ is compact and $C(K) > 0$, the F -functional of K is defined as in [11] by the formula

$$(2.3) \quad F(K) := \log C(K) - \int_K Q d\nu_K.$$

For admissible weight functions, we proved

THEOREM 2.2 [11]. *There exists a unique compact set $\mathfrak{S}_w \subset \Sigma \setminus Z$ with $C(\mathfrak{S}_w) > 0$ that has the following properties:*

(a) *For every compact set $K \subset \Sigma \setminus Z$ with $C(K) > 0$,*

$$(2.4) \quad F(K) \leq F(\mathfrak{S}_w)$$

where F is defined in (2.3).

(b) *If equality holds in (2.4), then $\mathfrak{S}_w \subset K$.*

(c) *For any positive integer n , if $P \in \Pi_n$ and the inequality*

$$(2.5) \quad |[w(x)]^n P(x)| \leq 1$$

holds q.e. on \mathfrak{S}_w , then it holds q.e. on Σ .

(d) *If Σ is regular, i.e. for all k large, $\Sigma \cap [-k, k]$ is regular with respect to the Dirichlet problem for its complement on the Riemann sphere, then for every $P \in \Pi_n$ and every $n = 1, 2, \dots$,*

$$(2.6) \quad \|[w(x)]^n P(x)\|_{\infty, \Sigma} = \|[w(x)]^n P(x)\|_{\infty, \mathfrak{S}_w},$$

where $\|\cdot\|_{\infty, A}$ denotes the sup norm over a set A .

(e) *In particular, when $\Sigma \setminus Z$ is a finite union of disjoint nondegenerate intervals and Q is convex in each of the components of $\Sigma \setminus Z$, then \mathfrak{S}_w is itself a finite union of nondegenerate disjoint closed intervals, at most one in each component of $\Sigma \setminus Z$; moreover, if $K \subset \Sigma \setminus Z$ is compact with $C(K) > 0$, then $F(K) < F(\mathfrak{S}_w)$ unless $\mathfrak{S}_w \subset K$ and $C(K \setminus \mathfrak{S}_w) = 0$.*

The major theorems of this paper can now be formulated as follows.

THEOREM 2.3. *Let w^λ be admissible for every $\lambda \in (0, 1]$, $n \geq 1$ be an integer and $P \in \Pi_n$. Suppose that*

$$(2.7) \quad |[w(x)]^n P(x)| \leq 1 \quad \text{q.e. on } \mathfrak{S}_w,$$

where \mathfrak{S}_w is given by Theorem 2.2. Then

$$(2.8) \quad |[w(x)]^n P(x)| \leq e^{-cn} < 1 \quad \text{q.e. on } \Sigma \setminus \mathfrak{S}_w;$$

where the constant $c := c(w, x) > 0$ is independent of n and P . Moreover, if Σ is regular, then there is a compact set $\mathfrak{S}^ \supset \mathfrak{S}_w$ with $C(\mathfrak{S}^* \setminus \mathfrak{S}_w) = 0$ such that for every compact set $K \subset \Sigma \setminus \mathfrak{S}^*$,*

$$(2.8a) \quad \|[w(x)]^n P(x)\|_{\infty, K} \leq e^{-cn} < 1,$$

where $c := c(w, K) > 0$ is independent of P and n .

We will show that the set \mathfrak{S}^* in Theorem 2.3 can be taken as $\mathfrak{S}^* = \bigcap_{n=1}^{\infty} \mathfrak{S}_{1/n}$, where $\mathfrak{S}_{1/n}$ is the extremal set of Theorem 2.2 corresponding to the weight $[w(x)]^{1/(1+\delta)}$, with $\delta = 1/n$ (see Lemma 3.4).

For our new results for L^p -norms, we need the following definitions.

DEFINITION 2.4. Let $E \subset \mathbf{R}$ be Lebesgue measurable. We say that E is interval-like if for every $c > 0$ there is a sequence $\{\delta_n\}$ of positive numbers (depending upon E and c) with the following properties:

- (i) $\delta_n \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) $\liminf \delta_n^{1/n} = 1$;
- (iii) For quasi-all $x \in E$,

$$m(E \cap I_n(x)) \geq (1 - c/n^2)\delta_n, \quad n \geq 1,$$

where, for each n , $I_n(x)$ is one of the intervals $[x, x + \delta_n]$, $[x - \delta_n, x]$, and m denotes the Lebesgue measure.

DEFINITION 2.5. We say that w is strongly admissible if

- (i) w^λ is admissible for every λ , $0 < \lambda \leq 1$,
- (ii) Σ is regular, and
- (iii) $\Sigma \setminus Z$ is interval-like.

If $A \subseteq \mathbf{R}$ is Lebesgue measurable, $g: \mathbf{R} \rightarrow \mathbf{R}$ is Lebesgue measurable, and $0 < p < \infty$, we set

$$(2.9) \quad \|g\|_{p,A} := \left(\int_A |g(x)|^p dx \right)^{1/p}.$$

For strongly admissible weight functions, the following theorem states that, in a sense, the L^p -norms of $w^n P$ also “live” on \mathfrak{S}_w .

THEOREM 2.6. Suppose that w is strongly admissible and $0 < p < \infty$.

(a) Let $\eta > 0$. Then there are constants $c_1 := c_1(w, \eta, p) > 0$, $c_2 := c_2(w, \eta, p) > 0$ and a compact set $\Delta := \Delta(w, \eta, p)$ with $m(\Delta) < \eta$ such that for every integer $n \geq 1$ and $P \in \Pi_n$,

$$(2.10) \quad \|w^n P\|_{p,\Sigma} \leq (1 + c_1 \exp(-c_2 n)) \|w^n P\|_{p,\mathfrak{S}_w \cup \Delta}.$$

(b) Let $0 < p, r \leq \infty$ and $\eta > 0$. Then, there exists a set $\Delta := \Delta(w, \eta, p, r)$ with $m(\Delta) < \eta$ such that whenever a polynomial $P \in \Pi_n$ satisfies

$$(2.11) \quad \|[w(x)]^n P(x)\|_{p,\mathfrak{S}_w \cup \Delta} \leq 1,$$

we have

$$(2.12) \quad \|[w(x)]^n P(x)\|_{r,\Sigma \setminus (\mathfrak{S}_w \cup \Delta)} \leq c_1 \exp(-c_2 n),$$

where $c_1 := c_1(w, \eta, p, r)$ and $c_2 := c_2(w, \eta, p, r)$ are positive constants independent of n and P .

THEOREM 2.7. Let $\Sigma \setminus Z$ be a finite union of nondegenerate disjoint intervals and Q be convex in each component of $\Sigma \setminus Z$. Assume that w is strongly admissible. Then $\mathfrak{S}_w =: \bigcup_{j=1}^l [a_j, b_j]$ (cf. Theorem 2.2(e)). Let $\{\varepsilon_j\}_{j=1}^l$ be arbitrary positive numbers.

(a) Then inequality (2.10) holds with $\mathfrak{S}_w \cup \Delta = \bigcup_{j=1}^l [a_j - \varepsilon_j, b_j + \varepsilon_j]$ for every $p > 0$. (The constants c_1, c_2 will now depend upon w, p and $\{\varepsilon_j\}_{j=1}^l$.)

(b) If $0 < p, r \leq \infty$, then with $\mathfrak{S}_w \cup \Delta = \bigcup_{j=1}^l [a_j - \varepsilon_j, b_j + \varepsilon_j]$, any polynomial $P \in \Pi_n$ that satisfies (2.11) also satisfies (2.12).

To illustrate the result of Theorem 2.7 we discuss the special case of an exponential weight on $[0, +\infty)$.

EXAMPLE. Let $w(x) := \exp(-x^\alpha)$, $\alpha > 0$, with support $\Sigma := [0, +\infty)$. Then $Z = \emptyset$ and $Q(x) = \log[1/w(x)] = x^\alpha$ is convex for $\alpha \geq 1$. Hence, by Theorem 2.2 (e), the set \mathfrak{S}_w for $\alpha \geq 1$ consists of a single compact interval $[a, b] \subset [0, +\infty)$. For $0 < \alpha < 1$, the function $Q(x)$ is no longer convex, but does possess the property that $xQ'(x)$ is increasing on $[0, +\infty)$. It is not difficult to show that this property again implies that \mathfrak{S}_w is a single compact interval. To explicitly determine \mathfrak{S}_w we consider the F -functional (cf. (2.3)) for intervals $K = [c, d] \subset [0, +\infty)$. Since

$$C([c, d]) = \frac{d-c}{4} \quad \text{and} \quad d\nu_{[c, d]} = \frac{1}{\pi} \frac{dx}{\sqrt{(d-x)(x-c)}},$$

we find

$$\begin{aligned} F([c, d]) &= \log \left(\frac{d-c}{4} \right) - \frac{1}{\pi} \int_c^d \frac{x^\alpha dx}{\sqrt{(d-x)(x-c)}} \\ &= \log \left(\frac{d-c}{4} \right) - \frac{1}{\pi 2^\alpha} \int_0^\pi [d+c - (d-c) \cos \theta]^\alpha d\theta. \end{aligned}$$

On computing the partial derivatives $\partial F/\partial c, \partial F/\partial d$, it is straightforward to show that F is maximized when $c = 0$ and

$$d = d_\alpha := \left[2\alpha\pi^{-1} \int_0^{\pi/2} \sin^{2\alpha} \theta d\theta \right]^{-1/\alpha} = \left[\frac{\sqrt{\pi}\Gamma(\alpha+1)}{\alpha\Gamma(\alpha+\frac{1}{2})} \right]^{1/\alpha}.$$

Hence $\mathfrak{S}_w = [0, d_\alpha]$.

As a consequence of Theorem 2.7(a), for each $p > 0$ and $\varepsilon > 0$, there exist positive constants c_1, c_2 depending on α, p , and ε such that for every $n \geq 1$ and $P \in \Pi_n$,

$$\int_0^\infty |e^{-nx^\alpha} P(x)|^p dx \leq (1 + c_1 \exp(-c_2 n))^p \int_0^{d_\alpha + \varepsilon} |e^{-nx^\alpha} P(x)|^p dx.$$

Furthermore, suppose that $P_n \in \Pi_n$, $n = 1, 2, \dots$, is a sequence of polynomials such that for some $p > 0$ and $\varepsilon > 0$

$$\int_0^{d_\alpha + \varepsilon} |e^{-nx^\alpha} P_n(x)|^p dx \leq 1, \quad n = 1, 2, \dots$$

Then, from Theorem 2.7(b) with $r = \infty$, we deduce that

$$e^{-nx^\alpha} P_n(x) \rightarrow 0 \quad \text{for all } x > d_\alpha.$$

For other applications of Theorem 2.7, see [8 and 12].

The proof of Theorem 2.3 uses potential theoretic arguments while the proofs of Theorems 2.6 and 2.7 utilize a general Nikolskii-type inequality (Lemma 3.7) in addition to Theorem 2.3. Using Nikolskii-type inequalities, we can also deduce asymptotic properties of certain extremal polynomials. These polynomials, in particular, include the orthogonal polynomials with respect to the Freud weights $\exp(-|x|^\alpha)$. Similar extremal problems have been studied by Gonchar and Rakhmanov [5].

In order to state our applications to polynomial extremal problems, we define

$$(2.13) \quad E_{n,p}(w) := \inf \{ \| [w(x)]^n [x^n - P(x)] \|_{\Sigma, p} : P \in \Pi_{n-1} \},$$

$n = 1, 2, \dots$, $0 < p \leq \infty$. The extremal polynomials $T_n(x; w, p)$ are defined by the properties

$$(2.14a) \quad T_n(x; w, p) = x^n + \dots \in \Pi_n,$$

$$(2.14b) \quad \|[w(x)]^n T_n(x; w, p)\|_{\Sigma, p} = E_{n,p}(w).$$

In particular, $T_n(x; w, 2)$ is the n th member of the system of monic orthogonal polynomials with respect to the weight function w^{2n} and $[E_{n,2}(w)]^{-1}T_n(x; w, 2)$ is the corresponding orthonormalized polynomial.

THEOREM 2.8. *Let w be strongly admissible. Then*

$$(2.15) \quad \lim_{n \rightarrow \infty} [E_{n,p}(w)]^{1/n} = \exp(F(\mathfrak{S}_w)); \quad 0 < p \leq \infty,$$

where \mathfrak{S}_w is defined in Theorem 2.2.

To illustrate Theorem 2.8 we again consider the weight $w = w_\alpha(x) := \exp(-x^\alpha)$, $\alpha > 0$, on $\Sigma = [0, +\infty)$. Referring to the example following Theorem 2.7, a simple computation yields

$$F(\mathfrak{S}_w) = F([0, d_\alpha]) = \log(d_\alpha/4) - 1/\alpha.$$

Hence, by Theorem 2.8, the minimal errors

$$E_{n,p}(w_\alpha) = \inf \left\{ \left[\int_0^\infty e^{-npx^\alpha} |x^n - P(x)|^p dx \right]^{1/p} : P \in \Pi_{n-1} \right\}$$

satisfy for each $p > 0$

$$\lim_{n \rightarrow \infty} [E_{n,p}(w_\alpha)]^{1/n} = \exp(F(\mathfrak{S}_w)) = d_\alpha/(4e^{1/\alpha}).$$

In order to describe the distribution of zeros of the extremal polynomials $T_n(x; w, p)$, we recall our previous results [11] concerning the solution of a generalized energy problem. Let $\mathfrak{M}(\Sigma)$ denote the collection of all positive unit Borel measures μ with $\text{supp}(\mu) \subset \Sigma$. For $\mu \in \mathfrak{M}(\Sigma)$, and $Q(x) = \log[1/w(x)]$, we put

$$(2.16) \quad I_w(\mu) := \iint [\log|x-t| - Q(x) - Q(t)] d\mu(x) d\mu(t).$$

Let

$$(2.17) \quad V_w := \sup\{I_w(\mu) : \mu \in \mathfrak{M}(\Sigma)\}.$$

We proved in [11] that V_w is a (finite) real number and that there exists a unique $\mu_w \in \mathfrak{M}(\Sigma)$ such that

$$(2.18) \quad V_w = I_w(\mu_w).$$

The measure μ_w was shown to be the limiting distribution of the zeros of $T_n(x; w, \infty)$ under certain conditions on w [11, Corollary 2.5]. Various other interesting properties of μ_w , also proved in [11] are summarized in Lemma 3.1. The following theorem is a generalization, in an L^p sense, of Theorem 2.4 of [11].

THEOREM 2.9. *Let w be strongly admissible. Suppose that $I \subset \mathbf{R}$ is a closed bounded interval containing \mathfrak{S}_w . Let $\{t_{k,n}\}_{k=1}^n$, $n = 1, 2, \dots$, be a triangular scheme of points lying in I . With this scheme, associate the sequence of polynomials*

$$q_n(x) := \prod_{k=1}^n (x - t_{k,n}), \quad n = 1, 2, \dots,$$

and the sequence of unit measures $\{\nu^{(n)}\}_{n=1}^\infty$, where for any Borel set \mathcal{B}

$$(2.19) \quad \nu^{(n)}(\mathcal{B}) := (1/n)|\{k: t_{k,n} \in \mathcal{B}\}|, \quad n = 1, 2, \dots$$

Assume that for some p ($0 < p \leq \infty$)

$$(2.20) \quad \limsup_{n \rightarrow \infty} \|w^n q_n\|_{\Sigma, p}^{1/n} \leq \exp(F(\mathfrak{S}_w)).$$

Then, in the weak-star topology,

$$(2.21) \quad \lim_{n \rightarrow \infty} \nu^{(n)} = \mu_w.$$

Moreover,

$$(2.22) \quad \lim_{n \rightarrow \infty} |q_n(z)|^{1/n} = \exp \left[\int \log |z - t| d\mu_w(t) \right],$$

uniformly on every compact set of the complex plane disjoint from I .

COROLLARY 2.10. *Let w be strongly admissible and $0 < p \leq \infty$. Let $\{t_{k,n}\}_{k=1}^n$ be the zeros of the extremal polynomial $T_n(x; w, p)$ of (2.14). Then there exists a closed bounded interval I containing \mathfrak{S}_w and all the zeros $\{t_{k,n}\}_{k=1}^n$, $n = 1, 2, \dots$. Moreover, the relations (2.21) and (2.22) hold with $q_n(z) = T_n(z; w, p)$.*

3. Proofs. Before providing the proofs of our theorems, we need to recall certain properties of the extremal measure μ_w defined in (2.18). These are summarized in Lemma 3.1. In the statement of this lemma and in the sequel, we assume, without loss of generality, that $Q(x) \geq 0$ for all $x \in \Sigma$.

LEMMA 3.1 [11]. *Let w be admissible. Then*

- (a) *The measure μ_w has finite logarithmic energy.*
- (b) *The set \mathfrak{S}_w of Theorem 2.2 is given by $\mathfrak{S}_w = \text{supp}(\mu_w)$.*
- (c) *The inequality*

$$(3.1) \quad \int \log |x - t| d\mu_w(t) \leq Q(x) + F(\mathfrak{S}_w)$$

holds q.e. on Σ .

- (d) *The inequality*

$$(3.2) \quad \int \log |x - t| d\mu_w(t) \geq Q(x) + F(\mathfrak{S}_w)$$

holds for all $x \in \mathfrak{S}_w$.

- (e) *The F -functional for \mathfrak{S}_w is given by*

$$(3.3) \quad F(\mathfrak{S}_w) = V_w + \int Q d\mu_w,$$

where V_w is defined in (2.17).

(f) For any positive integer n , if $P \in \Pi_n$ and

$$(3.4) \quad |[w(x)]^n P(x)| \leq 1 \quad \text{q.e. on } \mathfrak{S}_w,$$

then for all $z \in \mathbb{C}$

$$(3.5) \quad |P(z)| \leq \exp \left\{ n \left[\int \log |z - t| d\mu_w(t) - F(\mathfrak{S}_w) \right] \right\}.$$

(g) If Σ is regular, then (3.1) holds for all points of $\Sigma \setminus \mathfrak{S}_w$.

Assertions (a)–(f) are contained in Theorem 2.3 of [11] while part (g) is an observation in the proof of Theorem 2.1(c') in [11, p. 84].

In what follows, we shall assume that w^λ is admissible for every λ , $0 < \lambda \leq 1$. For brevity, we denote the extremal measure μ_w by μ , its support \mathfrak{S}_w by \mathfrak{S} and $F(\mathfrak{S})$ by F . Next, we define, for $\delta > 0$,

$$(3.6) \quad w_\delta(x) := \exp(-Q_\delta(x)) := \exp\left(-\frac{1}{1+\delta}Q(x)\right).$$

Since w_δ is admissible, we may apply our results in [11] to w_δ and get the corresponding extremal measure μ_δ with $\text{supp}(\mu_\delta) =: \mathfrak{S}_\delta$. Thus, \mathfrak{S}_δ will maximize the corresponding F -functional defined for compact K with $C(K) > 0$ by the formula

$$(3.7) \quad F_\delta(K) := \log C(K) - \int Q_\delta d\nu_K.$$

The quantity $F_\delta(\mathfrak{S}_\delta)$ will be denoted by F_δ .

The following two lemmas will play a central role in the proof of Theorem 2.3.

LEMMA 3.2. Suppose that μ is a nonnegative measure with finite logarithmic energy and ν is any measure with

$$(3.8) \quad \|\nu\| \leq \|\mu\|.$$

Then, if the inequality

$$(3.9) \quad \int \log |x - t| d\nu(t) \leq \int \log |x - t| d\mu(t) + c$$

holds μ -almost everywhere, it must hold everywhere in the complex plane \mathbb{C} .

Lemma 3.2 is a variant of the Second Maximum Principle. Landkof [6] gives a proof of this principle for the case of Riesz potentials. Analogous arguments for the logarithmic potential in the plane lead to the version stated in Lemma 3.2.

LEMMA 3.3. Let $\delta > 0$ and suppose that $x_0 \in \Sigma \setminus \mathfrak{S}_\delta$ satisfies

$$(3.10) \quad \int \log |x_0 - t| d\mu_\delta(t) \leq Q_\delta(x_0) + F_\delta.$$

Then

$$(3.11) \quad \int \log |x_0 - t| d\mu(t) < Q(x_0) + F.$$

PROOF. Since $x_0 \notin \mathfrak{S}_\delta$ and \mathfrak{S}_δ is compact, there exists a polynomial P such that

$$(3.12) \quad |P(x_0)| > 3/4 \quad \text{and} \quad |P(x)| < 1/4 \quad \text{for all } x \in \mathfrak{S}_\delta.$$

Let $r := N/\delta$, where N is the degree of P . Then (3.12) and Lemma 3.1(c) imply that

$$(3.13) \quad \log |P(x)| + r \int \log |x - t| d\mu(t) \leq rQ(x) + rF + \log(1/4) \quad \text{q.e. on } \mathfrak{S}_\delta.$$

Also, from Lemma 3.1(c), (d), we have

$$(3.14) \quad Q_\delta(x) = \int \log |x - t| d\mu_\delta(t) - F_\delta \quad \text{q.e. on } \mathfrak{S}_\delta.$$

Since $1 + \delta = (N + r)/r$, we see from (3.13) and (3.14) that quasi-everywhere on \mathfrak{S}_δ

$$(3.15) \quad \begin{aligned} & \log |P(x)| + r \int \log |x - t| d\mu(t) \\ & \leq (N + r) \int \log |x - t| d\mu_\delta(t) - (N + r)F_\delta + rF + \log(\tfrac{1}{4}). \end{aligned}$$

Note that, by Lemma 3.1(a), the measure μ_δ has finite logarithmic energy. Hence the maximum principle of Lemma 3.2 implies that (3.15) holds for all $x \in \mathbb{C}$. In particular, with $x = x_0$, we obtain from (3.15) and (3.10),

$$(3.16) \quad \log |P(x_0)| + r \left\{ \int \log |x_0 - t| d\mu(t) - Q(x_0) - F \right\} \leq \log(\tfrac{1}{4}).$$

Finally, since $\log |P(x_0)| > \log(\frac{3}{4})$, we see from (3.16) that

$$(3.17) \quad \int \log |x_0 - t| d\mu(t) - Q(x_0) - F \leq r^{-1} \log(\tfrac{1}{3}) < 0. \quad \square$$

In the next lemma, we summarize certain relationships between \mathfrak{S}_δ 's.

LEMMA 3.4. *Let $\mathfrak{S}^* := \bigcap_{n=1}^{\infty} \mathfrak{S}_{1/n}$. Then*

$$(3.18) \quad \mathfrak{S} \subset \mathfrak{S}^* \quad \text{and} \quad C(\mathfrak{S}^* \setminus \mathfrak{S}) = 0,$$

and

$$(3.19) \quad \lim_{n \rightarrow \infty} F_{1/n} = F, \quad \lim_{n \rightarrow \infty} \mu_{1/n} = \mu,$$

where the limit of the measures is the weak limit.

PROOF. Let

$$E_n := \{x_0 \in \Sigma : (3.10) \text{ does not hold with } \delta = 1/n\}, \quad n = 1, 2, \dots,$$

$$E_0 := \{x \in \Sigma : (3.1) \text{ does not hold}\},$$

$$E := E_0 \cup \left(\bigcup_{n=1}^{\infty} E_n \right).$$

Since each of the E_n 's and E_0 has capacity zero, it follows that $C(E) = 0$. Let $x \in \mathfrak{S} \setminus \mathfrak{S}^*$. Then (3.2) holds and $x \notin \mathfrak{S}_{1/N}$ for some N . If $x \notin E$, then Lemma 3.3 yields a contradiction to (3.2). Thus, $\mathfrak{S} \setminus \mathfrak{S}^* \subset E$ and so $C(\mathfrak{S} \setminus \mathfrak{S}^*) = 0$. A similar application of Lemma 3.3 to $Q_{1/n}$ in place of Q shows also that $C(\mathfrak{S}_{1/m} \setminus \mathfrak{S}_{1/n}) = 0$ if $m \geq n$.

Now, if K is any compact set with $C(K) > 0$, then

$$(3.20) \quad \begin{aligned} F_{1/n}(K) &= \log C(K) - \frac{n}{n+1} \int_K Q d\nu_K \\ &= \log C(K) - \int_K Q d\nu_K + \frac{1}{n+1} \int_K Q d\nu_K. \end{aligned}$$

In view of our assumption that $Q \geq 0$ on \mathbf{R} , we now see that $F_{1/n}(K) \geq F(K)$ for every compact set K with $C(K) > 0$. Hence (cf. Theorem 2.2(a))

$$(3.21) \quad F_{1/n} := F_{1/n}(\mathfrak{S}_{1/n}) \geq F_{1/n}(\mathfrak{S}) \geq F(\mathfrak{S}) =: F.$$

If $K \subset \mathfrak{S}_1$ and $Q(x) \leq M$ for $x \in \mathfrak{S}_1$, then (3.20) also shows that

$$(3.22) \quad F_{1/n}(K) - F(K) \leq M/(n+1).$$

Thus, since $F(\mathfrak{S}) \geq F(\mathfrak{S}_{1/n})$ and $C(\mathfrak{S}_{1/n} \setminus \mathfrak{S}_1) = 0$,

$$(3.23) \quad \begin{aligned} F_{1/n}(\mathfrak{S}_{1/n}) - F(\mathfrak{S}) &\leq F_{1/n}(\mathfrak{S}_{1/n}) - F(\mathfrak{S}_{1/n}) \\ &= F_{1/n}(\mathfrak{S}_{1/n} \cap \mathfrak{S}_1) - F(\mathfrak{S}_{1/n} \cap \mathfrak{S}_1) \leq M/(n+1). \end{aligned}$$

Inequalities (3.21) and (3.23) give the first part of (3.19).

Next, for $\delta > 0$, $\nu \in \mathfrak{M}(\Sigma)$ and $n = 1, 2, \dots$, we put

$$(3.24a) \quad I_\delta(\nu) := \iint [\log |x - t| - Q_\delta(x) - Q_\delta(t)] d\nu(x) d\nu(t),$$

$$(3.24b) \quad V_\delta := \sup \{I_\delta(\nu) : \nu \in \mathfrak{M}(\Sigma)\},$$

$$(3.24c) \quad I_0(\nu) := I_w(\nu), \quad V_0 := V_w.$$

Since $C(\mathfrak{S}_{1/n} \setminus \mathfrak{S}_1) = 0$ and each of the measures $\mu_{1/n}, \mu$ has finite logarithmic energy, we may assume that each of these measures is supported on \mathfrak{S}_1 . Moreover, on $\mathfrak{S}_1, 0 \leq Q(x) \leq M$. So,

$$(3.25) \quad \begin{aligned} V_{1/n} &\geq I_{1/n}(\mu) = \iint \left[\log |x - t| - \frac{n}{n+1} Q(x) - \frac{n}{n+1} Q(t) \right] d\mu(x) d\mu(t) \\ &\geq I_0(\mu) = V_0 \geq I_0(\mu_{1/n}) \\ &= I_{1/n}(\mu_{1/n}) - \frac{2}{n+1} \int Q d\mu_{1/n} \geq V_{1/n} - \frac{2M}{n+1}. \end{aligned}$$

Thus,

$$(3.26) \quad \lim_{n \rightarrow \infty} V_{1/n} = V_0.$$

We shall use this fact to show the second half of (3.19), concerning the weak limit of $\{\mu_{1/n}\}$. Using Helly's theorem, every subsequence of $\{\mu_{1/n}\}$ has a weakly convergent subsequence. Therefore, it suffices to show that if $\{\sigma_k\}$ is any weakly convergent subsequence of $\{\mu_{1/n}\}$ and $\lim_{k \rightarrow \infty} \sigma_k =: \sigma$ then $\sigma = \mu$. We may assume further that $\{\sigma_k \times \sigma_k\}$ converges to $\sigma \times \sigma$, and that σ is supported on \mathfrak{S}_1 . Suppose $\sigma_k =: \mu_{1/n_k}$ and $\varepsilon > 0$. Then for sufficiently large $R > 0$ and large k , we have

$$(3.27) \quad \begin{aligned} I_0(\sigma) &\geq \iint [\log_R |x - t| - Q(x) - Q(t)] d\sigma(x) d\sigma(t) - \varepsilon/4 \\ &\geq \iint [\log_R |x - t| - Q(x) - Q(t)] d\sigma_k(x) d\sigma_k(t) - \varepsilon/2 \\ &\geq \iint [\log |x - t| - Q_{1/n_k}(x) - Q_{1/n_k}(t)] d\sigma_k(x) d\sigma_k(t) - 3\varepsilon/4 \\ &= V_{1/n_k} - 3\varepsilon/4 \geq V_0 - \varepsilon, \end{aligned}$$

where $\log_R y := \max(\log y, -R)$, $y > 0$. Thus, $I_0(\sigma) \geq V_0$. But, the definition of V_0 then gives $I_0(\sigma) = V_0 = I_0(\mu)$. Since μ is the unique measure satisfying this equation, we have $\sigma = \mu$. This proves that $\mu_{1/n} \rightarrow \mu$ as $n \rightarrow \infty$.

Finally, we need to show that $\mathfrak{S} \subset \mathfrak{S}^*$. Since $\mu_{1/n} \rightarrow \mu$ as $n \rightarrow \infty$, it follows from Lemma 3.1(d) and the principle of descent [6, p. 62] that

$$\begin{aligned} \int \log |x - t| d\mu(t) &\geq \limsup_{n \rightarrow \infty} \int \log |x - t| d\mu_{1/n}(t) \\ (3.28) \quad &\geq \limsup_{n \rightarrow \infty} \left[\frac{n}{n+1} Q(x) + F_{1/n} \right] \\ &= Q(x) + F, \quad \text{for } x \in \mathfrak{S}. \end{aligned}$$

Next we integrate both sides of (3.28) with respect to $d\nu_{\mathfrak{S}^*}(x)$. Interchanging the order of integration and using the fact that $C(\mathfrak{S} \setminus \mathfrak{S}^*) = 0$, we see that $F(\mathfrak{S}^*) \geq F(\mathfrak{S})$. Theorem 2.2(b) then gives $\mathfrak{S} \subset \mathfrak{S}^*$. \square

PROOF OF THEOREM 2.3. Let \mathfrak{S}^* be defined as in Lemma 3.4 and assume that $P \in \Pi_n$ satisfies

$$(3.29) \quad |[w(x)]^n P(x)| \leq 1 \quad \text{q.e. on } \mathfrak{S}.$$

Then Lemma 3.1(f) gives, for $x \in \Sigma$,

$$(3.30) \quad |[w(x)]^n P(x)| \leq \exp \left\{ n \left[\int \log |x - t| d\mu(t) - Q(x) - F(\mathfrak{S}) \right] \right\}.$$

In view of Lemma 3.3, for quasi-all $x \in \Sigma \setminus \mathfrak{S}^*$ and hence, for quasi-all $x \in \Sigma \setminus \mathfrak{S}$, this gives

$$(3.31) \quad |[w(x)]^n P(x)| \leq e^{-cn} < 1, \quad c := c(w, x) > 0.$$

When Σ is regular, it follows from Lemma 3.1(g) and the continuity of the logarithmic potential that (3.31) holds for all x in any compact subset $K \subset \Sigma \setminus \mathfrak{S}^*$, with $c := c(K) > 0$ independent of x in K . \square

REMARK. When $\Sigma \setminus Z$ is a finite union of nondegenerate disjoint intervals and Q is convex in each component of $\Sigma \setminus Z$ then each of the sets $\mathfrak{S}_{1/n}$ is a finite union of nondegenerate disjoint intervals, at most one in each component of $\Sigma \setminus Z$. This, together with the fact that $\mathfrak{S} \subset \mathfrak{S}^*$ and $C(\mathfrak{S}^* \setminus \mathfrak{S}) = 0$ shows that in this important special case, $\mathfrak{S}^* = \mathfrak{S}$. This fact generalizes our earlier results in [9 and 10].

A major step in the proof of Theorem 2.6 is to obtain Nikolskii-type inequalities relating the various L^p -metrics of weighted polynomials. This, in turn, requires an estimation of Christoffel functions. When $\Sigma \setminus Z$ is a union of finitely many disjoint nondegenerate intervals, this is easily done using the now classical ideas of Freud in [3 or 4]. For the more general case we need the following lemma.

LEMMA 3.5. *Let $0 < p < \infty$. Then there exists a constant $A_1 > 0$ depending upon p alone with the following property: If $n \geq 1$, $P \in \Pi_n$, $B \subset [-1, 1]$ is measurable and*

$$(3.32) \quad m([-1, 1] \setminus B) \leq A_1/n^2,$$

then

$$(3.33) \quad \|P\|_{p, [-1, 1]} \leq 2\|P\|_{p, B}.$$

PROOF. Let

$$(3.34) \quad m_B(P, y) := m \{x \in B : |P(x)| > y\}.$$

Then it is well known [16, Vol. II, p. 112] that

$$(3.35a) \quad \|P\|_{p,B}^p = p \int_0^\infty y^{p-1} m_B(P, y) dy,$$

$$(3.35b) \quad \|P\|_{p,[-1,1]}^p = p \int_0^\infty y^{p-1} m_{[-1,1]}(P, y) dy.$$

But $m_{[-1,1]}(P, y) = 0$ if $y > \|P\| := \|P\|_{\infty,[-1,1]}$. So,

$$(3.36) \quad \begin{aligned} \|P\|_{p,[-1,1]}^p &= p \int_0^{\|P\|} y^{p-1} \{m_B(P, y) + m_{[-1,1] \setminus B}(P, y)\} dy \\ &\leq \|P\|_{p,B}^p + (A_1/n^2) \|P\|^p \end{aligned}$$

provided $m([-1, 1] \setminus B) \leq A_1/n^2$.

Now in view of Corollary 16 in [13, p. 114],

$$(3.37) \quad \|P\|^p \leq A_2 n^2 \|P\|_{p,[-1,1]}^p.$$

Thus, if we choose A_1 so that $0 < (1 - A_1 A_2)^{-1} < 2^p$, then (3.36) gives (3.33). \square

In the case when $\Sigma \setminus Z$ is interval-like, we can now obtain an estimation of the Christoffel functions.

LEMMA 3.6. *Let w be strongly admissible in the sense of Definition 2.5 and $\mathfrak{S} := \mathfrak{S}_w$ be the unique compact set of Theorem 2.2. Put*

$$(3.38) \quad \lambda_n(w^{2n}, x) := \min_{P \in \Pi_n} [P(x)]^{-2} \int_{\Sigma} [P(t)w^n(t)]^2 dt,$$

$$(3.39) \quad \omega(Q, \delta) := \max \{|Q(t) - Q(y)| : y \in \mathfrak{S}, t \in \Sigma, |y - t| \leq \delta\},$$

$$(3.40) \quad d := \inf \{|y - z| : y \in \mathfrak{S}, z \in Z\}.$$

Then, for n sufficiently large, we have for all $x \in \Sigma$

$$(3.41) \quad \lambda_n(w^{2n}, x) \geq c_0 \delta_n n^{-2} \exp \{-2n\omega(Q, \delta_n)\} [w(x)]^{2n},$$

where the sequence $\{\delta_n\}$ satisfies the conditions of Definition 2.4 with $E = \Sigma \setminus Z$ and $c = A_1/2$ with A_1 defined in Lemma 3.5.

PROOF. First, let $x \in \mathfrak{S}$ be such that the condition (iii) in Definition 2.4 holds with $A_1/2$ in place of c . Choose n so large that $\delta_n \leq d/2$. Then,

$$\lambda_n(w^{2n}, x) \geq \min_{P \in \Pi_n} [P(x)]^{-2} \int_{(\Sigma \setminus Z) \cap I_n(x)} [P(t)w^n(t)]^2 dt,$$

and so in view of Lemma 3.5, we have

$$(3.42) \quad \begin{aligned} \lambda_n(w^{2n}, x) &\geq c w^{2n}(x) \exp(-2n\omega(Q, \delta_n)) \cdot \min_{P \in \Pi_n} \left\{ [P(x)]^{-2} \int_{I_n(x)} P(t)^2 dt \right\} \\ &\geq c \delta_n w^{2n}(x) \exp(-2n\omega(Q, \delta_n)) \cdot \min_{R \in \Pi_n} \left\{ [R(0)]^{-2} \int_0^1 R(t)^2 dt \right\}. \end{aligned}$$

Using standard estimations for the Legendre polynomials, we then get

$$(3.43) \quad \lambda_n(w^{2n}, x) \geq c_0 \delta_n n^{-2} \exp(-2n\omega(Q, \delta_n)) w^{2n}(x).$$

Setting

$$(3.44) \quad M_n := [\delta_n n^{-2} \exp(-2n\omega(Q, \delta_n))]^{-1},$$

we have then proved

$$(3.45) \quad w^{2n}(x) \lambda_n^{-1}(w^{2n}, x) \leq c_0^{-1} \cdot M_n \quad \text{q.e. on } \mathfrak{S}.$$

Since $\lambda_n^{-1}(w^{2n}, x)$ is a polynomial of degree $2n$, inequality (3.45) holds everywhere on Σ in view of Theorem 2.2(d). \square

Using Lemma 3.6, we may now proceed exactly as in [9] to get the following inequalities.

LEMMA 3.7. *Let w be strongly admissible and M_n be as in (3.44), $0 < p < r \leq \infty$ and $P \in \Pi_n$. Then there exists a constant $c > 0$ independent of n and P such that*

$$(3.46) \quad \| [w(x)]^n P(x) \|_{r, \Sigma} \leq c \cdot M_n^{1/p-1/r} \| [w(x)]^n P(x) \|_{p, \Sigma}.$$

Using the fact (cf. Definition 2.4) that

$$(3.47) \quad \lim_{n \rightarrow \infty} M_n^{1/n} = 1$$

it is now easy to see that even if Σ is unbounded, the L^p -norm of a weighted polynomial on Σ almost “lives” on a fixed, compact interval. The following lemma makes this precise.

LEMMA 3.8. *Let w be strongly admissible and $0 < p < \infty$. Then there is a fixed compact interval J , and constants $c_1, c_2 > 0$, depending only on p, w and Σ with the following property:*

If $P \in \Pi_n$, then

$$(3.48) \quad \|w^n P\|_{p, \Sigma} \leq (1 + c_1 e^{-c_2 n}) \|w^n P\|_{p, J \cap \Sigma}.$$

PROOF. First, let δ be an integer such that $\delta > 1/p$ and choose A such that (cf. (3.6))

$$(3.49) \quad \mathfrak{S}_\delta \subset [-A, A].$$

Then for $x \in [-A, A] \cap \Sigma$

$$(3.50) \quad \begin{aligned} |x^{n\delta} w(x)^n P(x)| &\leq A^{n\delta} \|w(x)^n P(x)\|_{\infty, [-A, A] \cap \Sigma} \\ &\leq A^{n\delta} \|w(x)^n P(x)\|_{\infty, \Sigma} \\ &\leq A^{n\delta} c M_n^{1/p} \|w^n P\|_{p, \Sigma}. \end{aligned}$$

Here the last inequality follows from Lemma 3.7. Now, in view of (3.47), let $n \geq 2$ be so large that

$$(3.51) \quad M_n^{1/n} \leq 2^{\delta p}.$$

Then, on writing $x^{n\delta} P(x) w(x)^n = x^{n\delta} P(x) w_\delta(x)^{n(1+\delta)}$ and using Theorem 2.2(d) and (3.50), we see that

$$(3.52) \quad |P(x) w(x)^n| \leq c(2A)^{n\delta} \|w^n P\|_{p, \Sigma} \cdot |x|^{-n\delta}, \quad x \in \Sigma.$$

Thus, with $B := 4A$,

$$(3.53) \quad \int_{\Sigma \cap (\mathbf{R} \setminus [-B, B])} |P(x)[w(x)]^n|^p dx \leq 2c^p (2A)^{n\delta p} \|w^n P\|_{p, \Sigma}^p \int_B^\infty x^{-n\delta p} dx \\ \leq c 2^{-n\delta p} \cdot \|w^n P\|_{p, \Sigma}^p.$$

The estimate (3.53) implies (3.48) with $J := [-B, B]$. \square

Using (3.48) and Hölder's inequality, we can now extend the Nikolskii-type inequalities in Lemma 3.7. We formulate this in the following theorem.

THEOREM 3.9. *Let w be strongly admissible and M_n be defined as in (3.44). If $0 < p, r \leq \infty, n \geq 1$ is an integer and $P \in \Pi_n$, then*

$$(3.54) \quad \|w^n P\|_{p, \Sigma} \leq c \cdot M_n^{|1/p - 1/r|} \|w^n P\|_{r, \Sigma},$$

where $c > 0$ is a constant independent of n and P .

With the aid of Lemma 3.8, we are now able to prove Theorem 2.6.

PROOF OF THEOREM 2.6. First, using Lemma 3.4, we choose $\delta > 0$ so that

$$(3.55) \quad m(\mathfrak{S}_\delta \setminus \mathfrak{S}) < \eta/2.$$

Next, choose a bounded open set U such that

$$(3.56) \quad m(\overline{U} \setminus \mathfrak{S}) < \eta \quad \text{and} \quad U \supset \mathfrak{S}_\delta.$$

Then, with the interval J chosen as in Lemma 3.8, $J \setminus U$ is a compact set contained in $\mathbf{R} \setminus \mathfrak{S}_\delta \subset \mathbf{R} \setminus \mathfrak{S}^*$. Hence, Theorem 2.3 implies that there is a constant $L > 0$ such that

$$(3.57) \quad |[w(x)]^n P(x)| \leq \exp(-Ln) \|[w(t)]^n P(t)\|_{\infty, \Sigma}, \quad x \in (J \setminus U) \cap \Sigma.$$

Next, in view of (3.44), we may choose n so large that

$$(3.58) \quad M_n^{1/p} \leq \exp(Ln/2).$$

Then Lemma 3.7 yields

$$(3.59) \quad |[w(x)]^n P(x)| \leq c \exp(-Ln/2) \|w^n P\|_{p, \Sigma}, \quad x \in (J \setminus U) \cap \Sigma,$$

and hence

$$(3.60) \quad \|[w(x)]^n P(x)\|_{p, (J \setminus U) \cap \Sigma} \leq c(2B)^{1/p} \exp(-Ln/2) \|w^n P\|_{p, \Sigma}.$$

Theorem 2.6(a) now follows from (3.53) and (3.56). Theorem 2.6(b) is an easy corollary of Theorem 2.6(a) and Theorem 3.9. We omit the details of the proof. \square

For the proof of Theorem 2.7, we observe that under the hypothesis of that theorem, $\mathfrak{S}^* = \mathfrak{S}$ (cf. the remark following the proof of Theorem 2.3). Hence, using Lemma 3.4, we choose a suitable \mathfrak{S}_δ and \overline{U} in (3.55) and (3.56) of the form required for $\mathfrak{S} \cup \Delta$ in Theorem 2.7. The proof of Theorem 2.6 then also yields Theorem 2.7.

Theorem 2.8 follows from Theorem 3.9 and the extension of our Theorem 4.2 of [11] due to H. Stahl (cf. "The note added in proof" in [11]). Theorem 2.9 follows from Theorem 2.4 of [11] and Theorem 3.9.

PROOF OF COROLLARY 2.10. In view of Theorems 2.8 and 2.9, we need only show that the zeros of the polynomials $T_n(x) = T_n(x; w, p)$, $n = 1, 2, \dots$, are uniformly bounded. It is easy to see that all these zeros are real. Let X_n denote a zero of $T_n(x; w, p)$ having largest magnitude. By Lemma 3.8, there exists a compact interval $[a, b]$ with the property that for each n large and every $P \in \Pi_n$,

$$(3.61) \quad \|w^n P\|_{p, \Sigma} \leq 2 \|w^n P\|_{p, [a, b] \cap \Sigma}.$$

(The validity of (3.61) for the case $p = \infty$ is immediate from Theorem 2.2(d).)

Suppose now that $X_n > b$ (the case when $X_n < a$ is similar). Let

$$\hat{T}_n(x) := \frac{x - b}{x - X_n} T_n(x; w, p).$$

Then, for $x \in [a, b]$,

$$|\hat{T}_n(x)| \leq \frac{b - a}{X_n - b} |T_n(x; w, p)|.$$

Hence, from (3.61), we have for each n large

$$(3.62) \quad \begin{aligned} \|w^n \hat{T}_n\|_{p, \Sigma} &\leq 2 \|w^n \hat{T}_n\|_{p, [a, b] \cap \Sigma} \leq 2 \left(\frac{b - a}{X_n - b} \right) \|w^n T_n\|_{p, [a, b] \cap \Sigma} \\ &\leq 2 \left(\frac{b - a}{X_n - b} \right) \|w^n T_n\|_{p, \Sigma}. \end{aligned}$$

Thus, since T_n is extremal, inequality (3.62) implies that $1 \leq (2(b - a)/(X_n - b))$, that is, $X_n \leq 3b - 2a$ for all n large. \square

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REFERENCES

1. S. S. Bonan, *Weighted mean convergence of Lagrange interpolation*, Ph.D. Dissertation, Ohio State Univ., Columbus, Ohio, 1982.
2. G. Freud, *Orthogonal polynomials*, Pergamon Press, London, 1971.
3. —, *On polynomial approximation with respect to general weights*, Lecture Notes in Math., vol. 399, Springer-Verlag, Berlin and New York, 1974, pp. 149–179.
4. —, *On Markov-Berstein-type inequalities and their applications*, J. Approximation Theory **35** (1982), 203–213.
5. A. A. Gonchar and E. A. Rakhmanov, *The equilibrium measure and distribution of zeros of extremal polynomials*, Math. Sb. **125** (1984), 117–127.
6. N. S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, Berlin and New York, 1972.
7. D. S. Lubinsky, *A weighted polynomial inequality*, Proc. Amer. Math. Soc. **92** (1984), 263–267.
8. H. N. Mhaskar, *A rate of convergence theorem for expansions in Freud polynomials*, T. R. #86–08, Bowling Green State Univ., 1986.
9. H. N. Mhaskar and E. B. Saff, *Extremal problems for polynomials with exponential weights*, Trans. Amer. Math. Soc. **285** (1984), 203–234.
10. —, *Polynomials with Laguerre weights in L^p* , Lecture Notes in Math., vol. 1105, Springer-Verlag, Berlin and New York, 1984, pp. 511–523.
11. —, *Where does the sup norm of a weighted polynomial live? (A generalization of incomplete polynomials)*, Constr. Approx. **1** (1985), 71–91.
12. —, *A Weierstrass-type theorem for certain weighted polynomials*, Proc. Internat. Conf. on Approximation Theory, St. Johns, N.F., Canada, 1984 (to appear).

13. P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. No. 213 (1979).
14. M. Tsuji, *Potential theory in modern function theory*, 2nd ed., Chelsea, New York, 1958.
15. R. A. Zalik, *Some weighted polynomial inequalities*, J. Approximation Theory **41** (1984), 39–50.
16. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, Cambridge, 1977.

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